

On The Origin of Cusps in Stellar Systems

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ABSTRACT

An origin is sought for the ubiquity of cusps, both in computer simulations of halo formation in hierarchical clustering cosmogonies and in observations of galactic nuclei by the Hubble Space Telescope (HST). The encounters of merging clumps that built the galaxies can be described by the collisional Boltzmann equation. Using insights gained by studying the simpler Fokker-Planck equation, we show that there is a steady-state, self-consistent, cusped solution of the collisional Boltzmann equation corresponding to $\rho \sim r^{-4/3}$. This equilibrium is both stable and an attractor. It is the natural end-point of the diffusive encounters of an ensemble of equal mass clumps. The introduction of a mass spectrum weakens the mass density cusp. The spike in the luminosity density can be accentuated or softened, depending on the form of the mass-luminosity relation. Possible applications to the cusped nuclei of early-type galaxies are discussed.

Subject headings: galaxies: kinematics and dynamics – galaxies: nuclei – galaxies: elliptical and lenticular, cD – galaxies: formation

1. INTRODUCTION

Over the last years, two strands of research have made galactic astronomers more aware of the importance of cusps. First, Navarro, Frenk & White (1996a) have presented evidence for the existence of a universal law for galaxy halos, in which the mass density ρ is cusped and behaves like r^{-1} at small radii. This provides a reasonably accurate description of the density law for all halo masses in a range of cosmologies with different global parameters and power spectra (Navarro, Frenk & White 1996b). Fukushige & Makino (1996) have run simulations with larger numbers of particles and smaller softening. They report that the cusp is steeper than r^{-1} and that the central regions of the halo are expanding. Secondly, there is an extensive data-set of images of galactic nuclei taken with the HST by Lauer et al. (1995). The brilliance of the central nuclei is a consequence of a power-law rise in the light distribution right down to the very limits of the resolution. The data-set has been analysed by Gebhardt et al. (1996), who argue for a bimodal distribution of cusp slopes of early-type galaxies. Giant, radio-loud, early-type galaxies have a luminosity density ν that is typically cusped like $r^{-0.8}$, whereas normal or dwarf, radio-quiet, early-type galaxies have much steeper power-law profiles like $r^{-1.9}$. Hardly any early-type galaxies are uncusped. Of course, there is more than one mechanism for the formation of stellar cusps (e.g., Bahcall & Wolf 1976; Faber et al. 1996; Syer & White 1996). The aim of this paper is to point out another possibility and discuss its likely applications.

2. CUSPED STEADY-STATE SOLUTIONS OF THE FOKKER-PLANCK AND COLLISIONAL BOLTZMANN EQUATIONS

Ever since the visionary contribution of Toomre & Toomre (1972), the importance of merging and accretion in the assemblage of elliptical galaxies and the halos of spiral galaxies has been widely acknowledged. The building of the galaxies involves both violent

relaxation and the encounters or collisions of merging clumps. A complete description of such collisional dynamics is offered by Gilbert (1968), albeit at the price of some mathematical complexity. Gilbert showed that the collisional effects may be divided into two distinct processes, systematic acceleration from polarisation clouds (dynamical friction) and diffusion. It therefore seems a reasonable first approach to study the dynamics with the simpler Fokker-Planck equation, which includes the essential physical processes. This equation models the energy changes of the clumps as the cumulative effects of many weak binary encounters. This is probably invalid in the cosmogonic applications we have in mind. Nonetheless, we shall see that certain steady-states of the Fokker-Planck equation hold good for the collisional Boltzmann equation, which fully incorporates the effects of strong encounters.

For spherically symmetric, isotropic galaxies, the Fokker-Planck equation reads (e.g., Spitzer 1987; Theuns 1996)

$$\left(\frac{\partial N(E, t)}{\partial t}\right)_{\text{enc}} = -\frac{\partial}{\partial E}[N(E, t)D_1(E)] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[N(E, t)D_2(E)] \quad (1)$$

Here, $N(E, t)dE$ is the number of clumps with energy in the range E to $E + dE$ at time t . The first and second terms on the right-hand side describe the effects of dynamical friction and diffusion respectively. For a clump of mass M moving through a system of equal mass clumps, the diffusion coefficients D_1 and D_2 are defined as (e.g., Spitzer 1987; Theuns 1996)

$$D_1(E) = \langle \Delta E \rangle_V = 16\pi^2 G^2 M^2 \log \Lambda \left[\int_E^\infty f(E') dE' - \frac{1}{p(E)} \int_0^E f(E') p(E') dE' \right], \quad (2)$$

$$D_2(E) = \langle (\Delta E)^2 \rangle_V = 32\pi^2 G^2 M^2 \log \Lambda \left[\frac{1}{p(E)} \int_E^\infty q(E') f(E') dE' + \frac{q(E)}{p(E)} \int_0^E f(E') dE' \right]. \quad (3)$$

Here, angled brackets denote averages over all clumps of energy E within the accessible volume V . The $\log \Lambda$ term is the ‘Coulomb logarithm’, while f is the phase space number density of the clumps. The quantity $p(E)$ is the density of states (Spitzer 1987, chap. 2)

$$p(E) = 16\pi^2 \int_0^{r_{\text{max}}(E)} [2(E - \phi(r))]^{1/2} r^2 dr, \quad (4)$$

while $q(E)$ is the total phase space volume with energy less than E

$$q(E) = \frac{16\pi^2}{3} \int_0^{r_{\max}(E)} [2(E - \phi(r))]^{3/2} r^2 dr. \quad (5)$$

In both these definitions, $r_{\max}(E)$ is the maximum radial excursion possible for a clump of energy E , while ϕ is the gravitational potential.

Are there any cusped, steady-state solutions of the Fokker-Planck equation for our crude facsimile of a proto-galaxy built from merging clumps? Suppose the potential and density have the scale-free form

$$\phi \sim r^{-\beta}, \quad \rho \sim r^{-(\beta+2)}, \quad \beta \neq 0, \quad (6)$$

where β is a constant describing the severity of the cusp. The isotropic phase space distribution $f(E)$ and the density of states $p(E)$ behave like (e.g., Evans 1994)

$$f(E) = f_0 E^{2/\beta-1/2}, \quad p(E) = p_0 E^{1/2-3/\beta}, \quad (7)$$

where f_0 and p_0 are constants. So, the number of clumps $N(E)$ with energy in the range E to $E + dE$ goes like

$$N(E) = f(E)p(E) = f_0 p_0 E^{-1/\beta}. \quad (8)$$

It is now straightforward to deduce the diffusion coefficients D_1 and D_2 as

$$D_1(E) = 16\pi^2 G^2 M^2 f_0 \log \Lambda \frac{\beta(2+3\beta)}{(\beta+4)(1-\beta)} E^{1/2+2/\beta}, \quad (9)$$

$$D_2(E) = 64\pi^2 G^2 M^2 f_0 \log \Lambda \frac{\beta^2}{(\beta+4)(1-2\beta)} E^{3/2+2/\beta}. \quad (10)$$

Substituting into the Fokker-Planck equation (1), we find that a stellar cusp can be in a steady-state under the effects of encounters if

$$\left(\frac{\partial N(E, t)}{\partial t} \right)_{\text{enc}} = 8\pi^2 G^2 M^2 f_0^2 p_0 \log \Lambda \frac{\beta(3\beta+2)(\beta+2)}{(\beta+4)(1-\beta)(1-2\beta)} E^{1/\beta-1/2} = 0. \quad (11)$$

Note that the case $\beta = 0$ corresponds to an isothermal cusp with $\rho \sim r^{-2}$. This is a steady-state solution, but it requires a special treatment as the gravitational potential is logarithmic rather than a power of radius. The other physical solution corresponds to $\beta = -2/3$ or

$$\rho \sim r^{-4/3}. \quad (12)$$

This satisfies the steady-state Fokker-Planck equation in an interesting way. The dynamical friction and diffusive flux terms on the right-hand side of (1) separately vanish.

Now, the Fokker-Planck equation is derived from the full collisional Boltzmann equation (e.g., Spitzer 1987)

$$\left(\frac{\partial N(E, t)}{\partial t}\right)_{\text{enc}} = \sum_{r=1}^{\infty} \frac{(-1)^r}{\Gamma(r+1)} \frac{\partial^r}{\partial E^r} [N \langle (\Delta E)^r \rangle_V]. \quad (13)$$

under the assumption that the velocity changes produced by the encounters are small and so only the first two terms in the Taylor expansion matter. This is unlikely to be the case in our cosmogonic application. But, directly from scaling arguments, the diffusion coefficients must behave like

$$D_r = \langle (\Delta E)^r \rangle_V \sim E^{2/\beta + r - 1/2}. \quad (14)$$

When $\beta = -2/3$, it follows that $N \langle (\Delta E)^r \rangle_V \sim E^{r-2}$. This means that every term in the Taylor expansion – bar the first – is annihilated. But, the first term vanishes because $D_1 = 0$. So, we are led to an astonishing conclusion: the stellar cusp with $\rho \sim r^{-4/3}$ is a steady-state solution not just of the Fokker-Planck equation, but also of the full collisional Boltzmann equation. This vindicates our earlier claim that an analysis of the Fokker-Planck equation may be a good guide to the collisional Boltzmann equation. The time-scale on which the effects of Fokker-Planck evolution become observable is the two-body relaxation time, which can be longer than the age of the Universe for a giant galaxy. Again, though, let us stress that we are merely using the Fokker-Planck equation as a guide to the

harder-to-handle collisional Boltzmann equation. Large-angle scattering of clumps by potential fluctuations can drive evolution on a much shorter time-scale.

This simple treatment may be refined by introducing a distribution of masses of the clumps. Suppose the phase space number density of lumps with mass in the range M to $M + dM$ is (c.f. Young (1977))

$$f(E, M) \sim M^{X-1} \exp[-ME^{\alpha/(2+X)}], \quad (15)$$

where α and X are constants. This form of the distribution function is attractive because, once the integration over all the masses has been performed, the phase space mass density is again a power of the energy. As $X \rightarrow \infty$, the distribution function reduces to a simple power of the energy for a single-mass species. If the potential and density are of the scale-free power-law form (6), then integrating (15) over all masses and energies to obtain the mass density in configuration space implies

$$\alpha = \frac{2+X}{1+X} \frac{\beta-4}{2\beta}. \quad (16)$$

It is a straightforward, but lengthy, calculation to work out the new diffusion coefficients. We find that – for any value of the mass spectrum parameter X – there is a cusped, steady-state solution of both the Fokker-Planck and the collisional Boltzmann equations with

$$\rho \sim r^{-\frac{4}{3} + \frac{14}{3(2+3X)}}. \quad (17)$$

In the single species limit, $X \rightarrow \infty$ and we recover $\rho \sim r^{-4/3}$. The effect of broadening the mass distribution is to weaken the severity of the cusp. To compare our solutions with the HST data on early-type galaxies, the important quantity is not the mass density, but the luminosity density. Suppose the mass-luminosity relation behaves like $L(M) \propto M^p$, then the luminosity density is cusped like

$$\nu \sim r^{-\frac{4}{3} - \frac{7(3p-5)}{3(2+3X)}}. \quad (18)$$

If $p > 5/3$, then the effect of the distribution of masses is to make the luminosity density cusp steeper than $r^{-4/3}$. Notice, too, that when most of the mass in the cusp is composed of dark, heavy remnants, then the luminosity spike is less steep than $r^{-4/3}$. If there is a distribution of masses, then the heavier objects settle deeper in the cusp – and so the luminosity spike can be enhanced or diminished, according to whether the heavier objects are luminous or dark.

What are the merging clumps? One possibility is that the clumps are individual stars. This is fine, as the diffusion coefficients (2) and (3) are derived for an ensemble of point masses. Another possibility is that the clumps are bound clusters of stars. Now, though, the diffusion coefficients should strictly speaking be modified to take account of the extended structure of the clumps. A third possibility is to identify the clumps with gas clouds that have not yet formed stars. This also requires modification of the diffusion coefficients to allow for the dissipative nature of lossy encounters between gas clumps.

3. CUSP THERMODYNAMICS AND STABILITY

Our solutions are in dynamic – but not thermodynamic – equilibrium. This is obvious as the only possible thermodynamic equilibrium for a stellar system is the isothermal sphere. So, even though the mass flux vanishes in our Fokker-Planck solutions, there is a heat flux transmitted from the exterior of the model to the center of the cusp. Defining the thermodynamic temperature $T(E, M)$ as (e.g., Inagaki & Lynden-Bell 1990)

$$[kT(E, M)]^{-1} = -\frac{1}{M} \frac{\partial \log f(E, M)}{\partial E} \quad (19)$$

then a nice picture to have in mind is the conduction of heat from the hotter outsides to the colder center of the cusp. In general, the heat flux $S(E)$ through phase space volume

$q(E)$ is defined by (Hénon 1961; Lynden-Bell 1996)

$$\frac{1}{16\pi^2 G^2 M^3 \log \Lambda} \frac{dS}{dE} = f(E) \int_0^E f(E') \frac{dq}{dE'} dE' + \frac{df}{dE} \left[\int_0^E f(E') q(E') dE' + q(E) \int_E^\infty f(E') dE' \right]. \quad (20)$$

For power-law cusps of the form (6), we readily find

$$S(E) \sim E^{3/2+1/\beta}. \quad (21)$$

So, for the single-species Fokker-Planck solution with $\beta = -2/3$, there is a constant heat flux from the outside to the interior. It has magnitude

$$S = \frac{25\pi}{768\sqrt{2}} \log \Lambda \frac{M\phi_a^{3/2}}{a}, \quad (22)$$

where ϕ_a is the gravitational potential at some reference radius a . There is an evident analogy here with the earlier work of Bahcall & Wolf (1976), who considered the collisional relaxation of stars in the potential of a central massive black hole. They too found a steady state with a constant heat flux maintaining the equilibrium by having the hole absorb the influx of energy. To sustain the self-consistent cusp, an analogous sink is required. What are the physical causes of this sink of kinetic energy in the center of the self-consistent Fokker-Planck cusps? There are a number of possibilities. In cosmogonic applications, the disruption of binary or multiple lumps (such as weakly bound clusters) can provide an energy sink. Softly bound objects absorb energy and eventually disassociate according to Heggie's (1975) Law. This effect is likely to be more important in galaxy halo formation than, for example, in the present-day evolution of the globular clusters. In dense galactic nuclei, collisions should affect a large fraction of stars (Davies 1996). The disruption of stars through coalescence or collision again provides a sink of kinetic energy – both in the direct impact of stars and via the raising of tides. This, though, is a delicate matter. If the density is high enough for collisions to be important, then hard binaries will also form. They will act as a heat source, counteracting the effects of collisions.

Is the self-consistent cusp stable? The evidence for stability is the series of numerical computations of the evolution of the Fokker-Planck equation that we have performed and will present elsewhere. The cusp remains unchanged over time-scales longer than the age of the Universe. When the Fokker-Planck equation is evolved with arbitrary initial conditions, the distribution function moves towards the steady-state cusp, which is therefore an attractor. These properties are analogous to those found by Bahcall & Wolf (1976) for cusped star distributions around a black hole. At first glance, though, they seem in contradiction with the numerical results of Quinlan (1996), who found that the double-power-law models evolve quickly (by expanding) when the inner cusp is $\rho \sim r^{-4/3}$. But, Quinlan’s simulations do not incorporate a central heat sink and so his interesting results are not directly applicable to our problem.

4. DISCUSSION

Navarro, Frenk & White (1996a,b) have recently argued that dark halos possess a universal density profile of the form

$$\rho \sim \frac{1}{r(a+r)^2} \quad (23)$$

In the inner parts of their simulations, the dynamics is driven by encounters. There is quite a lot of scatter in the density profile in the inner region of the simulations, which may still be affected by numerical limitations. So, the inner cusp profile could be $\rho \sim r^{-4/3}$ without violating the simulation data (White, private communication). This idea receives support from the higher-resolution simulations of Fukushige & Makino (1996), who find both that the cusp is steeper than $\rho \sim r^{-1}$ and that the central regions of the haloes are expanding. This is exactly what we would predict on theoretical grounds. Heat has to be transferred from the outer to the inner parts of the N-body simulation, which cannot be in exact thermodynamic equilibrium. In the absence of an energy sink, the central regions

must respond to the heat input by expansion. The intrinsic velocity dispersion of our cusp solutions falls on moving deeper into the cusp. This property also appears to be replicated in the simulations (White, private communication; Fukushige & Makino 1996).

Faber et al. (1996) argue that the shallow cusps of giant ellipticals may be caused by black hole binaries, but that the steep cusps of normal or dwarf ellipticals may be the result of the engulfment of gas-rich, small disk galaxies. Here, we raise the question: is it possible that some of the cusps correspond to these new steady-state, attracting solutions? This is difficult to establish for certain – as the light cusp depends on both the mass spectrum and the mass-luminosity relation. These are poorly known for the stars in the cusps. If the light in the cusps is dominated by main sequence stars, then a fair value for p is ~ 3 so that $L \propto M^3$. Then, the observed luminosity density spike of the radio-quiet early type galaxies can be reproduced if $X \sim 4$, which corresponds to a broadish mass spectrum. The mass density cusp is just $\rho \sim r^{-1}$ and the intrinsic velocity dispersion $\langle v^2 \rangle$ vanishes as $r \rightarrow 0$. Assuming the density cusp is matched to the outer parts of a model like (23), then the line-of-sight velocity dispersion falls logarithmically to zero at the centre. Does the kinematic evidence support this suggestion? The available ground-based data seem to show the line-of-sight velocity dispersion is constant or rising in the cusped nuclei of the sample (Faber et al. 1996). But, the inference of the velocity dispersion is a delicate matter in the centers of galactic nuclei where the line profile is non-Gaussian. If mass segregation of a stellar population operates to build the luminosity density cusp, then color gradients will be established. The available data is again scanty, but the general belief is that large color gradients are not present (Carollo et al. 1997). This, though, is not a serious objection. It is easy to evade the existence of colour gradients with more complex mass-luminosity relations than the simple power-laws – for example, if the cusp is built from a uniform stellar population and lower-mass black holes or dark remnants, mass segregation can build the observed luminosity spike, but no color gradients will be detected. Alternatively, if the

constituents of the cusp are a uniform stellar population and higher mass black holes, then the spike in the luminosity density will be less than $\nu \sim r^{-4/3}$ with no color gradients.

The main result of this paper is the demonstration of a new cusped family of self-consistent solutions to the Fokker-Planck and collisional Boltzmann equations. These are a natural end-point of the gravitational scattering of point masses in the presence of a central energy sink. Central light cusps are possible in the basin of a potential well without central massive objects.

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